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Harmonic oscillator field theory: I. Classical and semiclassical large-N limit

Ahmed Abouelsaood†‖ and Ashraf Abuelseoud‡

† Physics Department, Faculty of Science, Arab Emirates University, POB 17551 Alain, United Arab Emirates

‡ Physics Department, Northeastern University, Boston, MA, USA

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Abstract. A field theoretical formulation of the rather easy problem of N identical, one-dimensional bosons, interacting pairwise via harmonic oscillator potentials of equal force constants is given, in a way similar to the way the nonlinear Schrodinger model provides a field theoretical formulation of a one-dimentional, δ-function Bose gas. The nonlinear integro-differential equation obtained for the complex scalar field of the model is treated classically, and solved in the semiclassical limit of a large number of particles. Working in the centre of mass frame of the field, a sequence of operators that would raise or lower the energy of a state by definite amounts without changing its particle number are constructed from certain bilinear expressions in the field. These operators have, in the limit of a large number of particles, commutation relations similar to those of the Fourier modes of a compact free field, except that the first mode is missing as a result of the fact that the field in the centre of mass frame is subject to the obvious constraint that its centre of mass always coincides with the coordinate origin. The quantum states can be labelled by the total centre of mass momentum, and the quasi-particle numbers in each of the independent oscillator modes. The energy spectrum of the system is calculated, giving the same results as the exact, quantum-mechanical N-body analysis.

1. Introduction

The problem of N identical bosons interacting pairwise via harmonic oscillator potentials of equal force constants is exactly solvable both classically and quantum-mechanically in any number of dimensions [1]. In the one-dimensional case the Hamiltonian has the form

$$H = \sum_{i=1}^{N} \frac{P_i^2}{2\mu} + 2c \sum_{i,j=1}^{N} \left(x_i - x_j\right)^2 \tag{1}$$

where we have set $\hbar = 1$.

We can go to normal modes by means of the Jacobi coordinates

$$\xi_k = \frac{1}{\sqrt{k(k+1)}} \left(\sum_{n=1}^{k} x_n - n x_{n+1} \right) \qquad \text{for } k = 1, 2, \ldots, N-1$$

and

$$\xi_N = \frac{1}{N} \sum_{n=1}^{N} x_n \tag{2}$$

‖ On leave of absence from: Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Giza, Egypt.

in terms of which the Hamiltonian takes the form

$$H = \sum_{i=1}^{N-1} \left(\frac{\eta_i^2}{2\mu} + 2cN\xi_i^2 \right) + \frac{P_N^2}{2(N\mu)} \tag{3}$$

where η_i is the momentum conjugate of ξ_i.

Hence the Hamiltonian separates into the sum of the Hamiltonians of $N-1$ identical harmonic oscillators of mass μ and angular frequency $\omega_N = \sqrt{4cN/\mu}$, plus the centre of mass energy which equals the square of the centre of mass momentum P_N divided by twice the the total mass $N\mu$.

This problem however has a non-trivial part, which is to find what constraints the Bose symmetry would impose on the spectrum of the quantum theory. It turns out that the allowed energy values and their precise degrees of degeneracy are given by the formula

$$E(P_N; n_2, n_3, \ldots, n_N) = E_0 + \frac{P_N^2}{2N\mu} + \omega_N \sum_{k=2}^{N} k n_k \tag{4}$$

where E_0 is the zero point energy of the $N-1$ oscillators, and is given by

$$E_0 = \frac{(N-1)\omega_N}{2} \tag{5}$$

and n_k are a set of non-negative integers (the occupation numbers of the harmonic oscillator levels), which, when taken along with the centre of mass momentum P_N, completely specify the N-particle quantum state. Observe that in this formula the occupation number of the first level does not appear.

Although the complete solution sketched above for the N-particle problem has been known for a long time, it may still be of some interest to be able to derive the above results from a field-theoretical point of view, just in the same way as the problem of N one-dimensional bosons interacting pairwise via δ-function potentials can be solved by applying the quantum inverse scattering method to the nonlinear Schrödinger model [2], thus retrieving the well-known results about the N-body problem. It is the purpose of this series of papers to do this. The first paper will deal exclusively with the classical and semiclassical Theory. A fully-quantum treatment of the same problem will then follow. The goal will be to derive equation (4) using field-theoretical methods.

The Hamiltonian of the $(1+1)$-dimensional non-relativistic field theory corresponding to the many-body problem (1) with the mass μ set equal to $\frac{1}{2}$ is given by

$$H = \int_{-\infty}^{\infty} (\partial\phi^*)(\partial\phi)\,\mathrm{d}x + c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi^*(x')\phi(x')(x'-x)^2\phi^*(x)\phi(x)\,\mathrm{d}x\,\mathrm{d}x' \tag{6}$$

with the complex scalar field ϕ obeying the following equal-time commutation relations

$$[\phi(x,t), \phi(x',t)] = 0 = [\phi^*(x,t), \phi^*(x',t)] \qquad [\phi(x,t), \phi^*(x',t)] = \delta(x-x')\,. \tag{7}$$

The organization of this paper is as follows. In section 2, the classical field theory will be considered, and the most obvious symmetries and conservation laws will be discussed. In section 3, a method for solving the initial value problem for the field equation will be introduced. This method is based on making a series of field transformations that reduce the field equation to a quasi-linear form that can be dealt with using the ordinary techniques of linear analysis. In section 4, the canonical structure of the classical model will be studied. The difficulties of quantizing the system will be outlined in section 5, and a semiclassical approach valid for a large number of particles is suggested. The excitation operators are introduced in section 6. The semiclassical large-N limit is considered in section 7, and the low-lying states are constructed, thus obtaining equation (4) for the energy spectrum.

2. Symmetries and conservation laws

In the classical theory the field ϕ is a c-number complex scalar field obeying the Poisson bracket analogue of the commutators (7)

$$\mathrm{i}\left\{\phi(x,t),\phi(x',t)\right\} = 0 = \mathrm{i}\left\{\phi^*(x,t),\phi^*(x',t)\right\} \qquad \mathrm{i}\left\{\phi(x,t),\phi^*(x',t)\right\} = \delta(x-x')\,. \tag{8}$$

Using equations (6) and (8) we obtain the classical field equation

$$\mathrm{i}\partial_t\phi(x,t) = -\partial_x^2\phi(x,t) + 2c\phi(x,t)\int \mathrm{d}x'\,(x-x')^2\phi^*(x',t)\phi(x',t) \tag{9}$$

For any function $\hat{A}$ of x and ∂_x, it is useful to define the quantity

$$\left\langle\hat{A}(x,\partial_x)\right\rangle_\phi = \int \phi^*(x,t)\hat{A}(x,\partial_x)\phi(x,t) \tag{10}$$

just as in ordinary quantum mechanics.

The theory has the following obvious constants of motion:

(1) The total charge (particle number)

$$N = \int \phi^*(x,t)\phi(x,t)\,\mathrm{d}x = \langle 1\rangle_\phi \tag{11}$$

(2) The centre of mass momentum per particle

$$p_0 = \frac{\langle -\mathrm{i}\partial_x\rangle_\phi}{N} \tag{12}$$

(3) The total energy (or Hamiltonian)

$$H = \int(\partial_x\phi^*)(\partial_x\phi)\mathrm{d}x + c\int\!\!\int \phi^*\phi(x')(x-x')^2\phi^*\phi(x)\,\mathrm{d}x\,\mathrm{d}x'$$

which may be written in the form

$$H = \left\langle -\partial_x^2 + \tfrac{1}{4}\omega^2x^2\right\rangle_\phi - \tfrac{1}{4}\omega^2\,\langle x\rangle_\phi^2 \tag{13}$$

where $\omega^2 = 8cN$. This reflects the fact that the model is invariant under global gauge transformations, spatial translations and time translations respectively.

A fourth conserved quantity is the Casimir operator for the well known harmonic oscillator SL(2) algebra [3] generated by

$$\chi_+ = \tfrac{1}{2}\left\langle a^2\right\rangle_\phi \qquad \chi_- = \tfrac{1}{2}\left\langle (a^\dagger)^2\right\rangle_\phi \qquad \chi_0 = \tfrac{1}{2}\left\langle a^\dagger a + \tfrac{1}{2}\right\rangle_\phi \tag{14}$$

where

$$a = \frac{\sqrt{\omega}}{2}\left(x + \frac{2}{\omega}\partial_x\right) \qquad a^\dagger = \frac{\sqrt{\omega}}{2}\left(x - \frac{2}{\omega}\partial_x\right) \tag{15}$$

This algebra has the form

$$\mathrm{i}\left\{\chi_+,\chi_0\right\} = \chi_+ \qquad \mathrm{i}\left\{\chi_-,\chi_0\right\} = -\chi_- \qquad \mathrm{i}\left\{\chi_+,\chi_-\right\} = 2\chi_0\,. \tag{16}$$

The Casimir operator in this case is given by

$$\chi = \chi_3^2 - \chi_+\chi_- \tag{17}$$

and has zero Poisson brackets with the three other conserved quantities N, p_0 and H, and hence is also conserved.

3. Solving the initial value problem

The initial value problem for equation (9) consists in finding solutions of this equation satisfying the initial conditions

$$\phi(x, t=0) = \phi_0(x) \tag{18}$$

where $\phi_0(x)$ is some given function of x. We shall also require the boundary conditions

$$\lim_{x\to\pm\infty} \phi(x, t) = 0 \tag{19}$$

which should also be satisfied by the initial data. The method used here is to make a number of field transformations that will reduce the nonlinear partial integro-differential equation (9) into a linear problem that can be solved using the ordinary techniques (e.g. mode expansion, Green functions, etc), then by inverting the linearizing transformations we obtain the required solution to the nonlinear problem. This program parallels the inverse-scattering method program [4–6] for solving some nonlinear partial differential equations, but the details are quite different, particularly since this field theory, due to the confining quadratic potential, has no elementary particle scattering states, but only solitons.

The linearization process consists of three steps:

(1) Going to the centre of mass frame. The centre of mass coordinate x_0 is defined as

$$x_0(t) = \langle x\rangle_\phi / N\,. \tag{20}$$

One can easily see that it moves with a constant velocity $v_0 = 2p_0$, where p_0, the centre of mass momentum per particle, is given by equation (11), which means that

$$x_0(t) = x_0(t=0) + 2p_0 t\,. \tag{21}$$

The centre of mass field ϕ_1 is related to the original field ϕ by a combined spatial translation and Galilean transformation in the form

$$\phi_1(x, t) = \exp\{-\mathrm{i}[p_0(x + x_0) - p_0^2 t]\}\phi(x + x_0, t)\,. \tag{22}$$

It can be easily seen that the centre of mass of ϕ_1 coincides with the coordinate origin, which means that

$$\langle x\rangle_{\phi_1} = 0 \qquad \langle -\mathrm{i}\partial_x\rangle_{\phi_1} = 0\,. \tag{23}$$

From equations (9) and (22) it can be easily seen that ϕ_1 satisfies the field equation

$$\mathrm{i}\partial_t\phi_1(x, t) = \left[p_0^2 - \partial_x^2 + \tfrac{1}{4}\omega^2 x^2 + 2c\left\langle x^2\right\rangle_{\phi_1}\right]\phi_1(x, t)\,. \tag{24}$$

In terms of ϕ_1 the Hamiltonian has the form

$$H = p_0^2 N + \left\langle -\partial_x^2 + \tfrac{1}{4}\omega^2 x^2\right\rangle_{\phi_1}\,. \tag{25}$$

(2) Elimination of the $\langle x^2\rangle$ term. Although the transformation (22) has simplified to some extent the field equation (9), the resulting equation (25) is still nonlinear since the quantities ω^2 and $\langle x^2\rangle_{\phi_1}$ are quadratic functionals of the field ϕ_1. The first quantity ω^2 does not pose any problem since it is a constant of motion just as the factor p_0^2.

To eliminate the $\langle x^2\rangle_{\phi_1}$ term we define a new field ϕ_2 by the relation

$$\phi_2(x,t) = \exp[-(1/4N)(\tilde{\chi}_+ - \tilde{\chi}_-)]\phi_1(x,t) \tag{26}$$

where

$$\tilde{\chi}_+ = \left\langle \tfrac{1}{2}a^2 \right\rangle_{\phi_1} = \chi_+ - \frac{\omega}{8}\left(x_0 + \frac{\mathrm{i}2p_0}{\omega}\right)^2$$

$$\tilde{\chi}_- = \left\langle \tfrac{1}{2}\left(a^\dagger\right)^2 \right\rangle_{\phi_1} = \chi_- - \frac{\omega}{8}\left(x_0 - \frac{\mathrm{i}2p_0}{\omega}\right)^2$$

$$\tilde{\chi}_0 = \tfrac{1}{2}\left\langle a^\dagger a + \tfrac{1}{2}\right\rangle_{\phi_1}$$

are the generators for the ϕ_1 field which obey the same algebra (16) as those of the original ϕ field. One can verify that the field ϕ_2 satisfies the equation

$$\mathrm{i}\partial_t\phi_2(x,t) = \left[\frac{1}{2}\left(p_0^2 + \frac{H}{N}\right) - \partial_x^2 + \frac{1}{4}\omega^2x^2\right]\phi_2(x,t)\,. \tag{27}$$

(3) Factorizing the N-dependence. Equation (27) is almost linear, since all nonlinearity arises from the conserved multiplicative factors p_0^2, N and H. So this equation lends itself to the ordinary techniques used to solve linear differential equations. It is however advantageous from a canonical point of view, with an eye on the quantum theory, to make a scaling transformation to partially factor out the N-dependence of ω^2. So we define a new field ϕ_3 by the relation

$$\phi_3(x,t) = N^{1/8}\phi_2\left(\frac{x}{N^{1/4}}, t\right)\,. \tag{28}$$

This field obeys the equation

$$\mathrm{i}\partial_t\phi_3 = \left[\frac{1}{2}\left(p_0^2 + \frac{H}{N}\right) + \sqrt{N}\left(-\partial_x^2 + 2cx^2\right)\right]\phi_3 \tag{29}$$

It can be easily seen that the Hamiltonian can be written in terms of the ϕ_3 field as

$$H = p_0^2N + \sqrt{N}\left\langle -\partial_x^2 + 2cx^2\right\rangle_{\phi_3} \tag{30}$$

while the SL(2) generators are the same as those for ϕ_1 and ϕ_2.

The general solution of equation (29) can be written in terms of the Fourier modes of the field ϕ_3 as follows:

$$\phi_3(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x) \tag{31}$$

where $\psi_n(x)$ are the energy eigenfunctions of a harmonic oscillator of mass $\mu = \frac{1}{2}$ and angular frequency

$$\omega_0 = \sqrt{8c} = \omega/\sqrt{N}\,.$$

Substituting equation (31) in the field equation (27) for ϕ_3, we obtain

$$\mathrm{i}\partial_t a_n = \left[\frac{1}{2}\left(p_0^2 + \frac{H}{N}\right) + \left(n + \frac{1}{2}\right)\omega\right]a_n \tag{32}$$

which has the obvious solution

$$a_n(t) = a_n\,(t=0)\exp\left\{-\mathrm{i}\left[\frac{1}{2}\left(p_0^2 + \frac{H}{N}\right) + \left(n + \frac{1}{2}\right)\omega\right]t\right\}. \tag{33}$$

4. Canonical structure of the model

The next task would be to consider the quantum theory. As a first step towards this, Poisson brackets for the quasi-linear field ϕ_3 and its Fourier modes should be derived. This is quite a straightforward, though a somewhat tedious calculation in which the general properties of Poisson brackets are used repeatedly, in particular the obvious relation

$$\{\phi(x+A), B\} = \{\phi(x), A\}_{x\to x+A} + \{A, B\}\,\partial_x\phi(x+A) \tag{34}$$

where A and B are any two functions of the canonical variables. The details of the derivation will be omitted here since they are not particularly interesting, and only Poisson brackets for the field ϕ_3 will be listed below.

$$\mathrm{i}\left\{\phi_3(x,t), \phi_3(x',t)\right\} = (1/N)\left(x\partial_{x'} - x'\partial_x\right)\phi_3(x,t)\phi_3(x',t) \tag{35}$$

$$\mathrm{i}\left\{\phi_3(x,t), \phi_3^*(x',t)\right\} = \delta(x-x') + (1/N)\left(x\partial_{x'} + x'\partial_x\right)\phi_3(x,t)\phi_3^*(x',t)\,. \tag{36}$$

The above results have a very simple interpretation. We recall that the field ϕ_3 is subject to two constraints

$$\langle x\rangle_{\phi_3} = 0 \qquad\qquad \langle -\mathrm{i}\partial_x\rangle_{\phi_3} = 0\,. \tag{37}$$

According to Dirac's classification [7], these constraints are called second class since their Poisson bracket

$$\left\{\langle x\rangle_{\phi_3}, \langle -\mathrm{i}\partial_x\rangle_{\phi_3}\right\} = N \tag{38}$$

is non-zero, and is not a linear combination of other constraints. In systems with such constraints, Poisson brackets of the fields should be replaced by Dirac brackets in order to be consistent with the constraint equations. For any two dynamical variables A and B, Dirac bracket in our case is given by

$$\{A, B\}_D = \{A, B\} + \left\{A, \langle x\rangle_{\phi_3}\right\}\frac{1}{N}\left\{\langle -\mathrm{i}\partial_x\rangle_{\phi_3}, B\right\} - \left\{A, \langle -\mathrm{i}\partial_x\rangle_{\phi 3}\right\}\frac{1}{N}\left\{\langle x\rangle_{\phi_3}, B\right\} \tag{39}$$

where $\langle x\rangle_{\phi_3}$ and $\langle -\mathrm{i}\partial_x\rangle_{\phi_3}$ are set equal to zero only after evaluating their Poisson brackets considering ϕ_3 and ϕ_3^* to obey the same equal-time Poisson brackets as ϕ and ϕ^* (equation (8)). This gives precisely equations (35) and (36) as Dirac brackets for ϕ_3 and ϕ_3^*.

Considering now the Fourier modes a_n of the field ϕ_3 as defined by equation (31), we can easily see that the two constraints (37) are equivalent to the complex constraint

$$\sum_{m=0}^{\infty}\sqrt{m+1}\,a_{m+1}^*a_m = 0. \tag{40}$$

A direct computation shows that Poisson brackets for these modes are given by

$$\mathrm{i}\{a_m, a_n\} = \frac{1}{N}\left[\sqrt{m(n+1)}\,a_{m-1}a_{n+1} - \sqrt{n(m+1)}\,a_{m+1}a_{n-1}\right] \tag{41}$$

$$\mathrm{i}\left\{a_m, a_n^*\right\} = \delta_{m,n} + \frac{1}{N}\left[\sqrt{(m+1)(n+1)}\,a_{m+1}a_{n+1}^* - \sqrt{mn}\,a_{m-1}a_{n-1}^*\right]. \tag{42}$$

These Poisson brackets also follow as Dirac brackets for the Fourier modes of a scalar field constrained by equation (40). It is this point of view that will prove most fruitful when we come to discuss the quantum theory.

In terms of the Fourier modes of the field ϕ_3 we have

$$N = \sum_{n=0}^{\infty} a_n^* a_n \tag{43}$$

$$H = p_0^2 N + \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \omega a_n^* a_n \tag{44}$$

$$\begin{aligned} \tilde{\chi}_0 &= \frac{1}{2}\left\langle a^\dagger a + \frac{1}{2}\right\rangle_{\phi_3} = \frac{1}{2}\sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right) a_n^* a_n \\ \tilde{\chi}_+ &= \frac{1}{2}\sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}\, a_n^* a_{n+2} = \tilde{\chi}_-^* . \end{aligned} \tag{45}$$

The following Poisson brackets can easily be derived using equations (41)–(45):

$$\mathrm{i}\{a_m, H\} = \tfrac{1}{2}\left(m + \tfrac{1}{2}\right)\omega a_m \qquad \mathrm{i}\{a_m, N\} = a_m \tag{46}$$

$$\mathrm{i}\left\{a_m, \tilde{\chi}_+\right\} = \tfrac{1}{2}\sqrt{(m+1)(m+2)}\, a_{m+2} \tag{47}$$

$$\mathrm{i}\left\{a_m, \tilde{\chi}_-\right\} = \tfrac{1}{2}\sqrt{m(m-1)}\, a_{m-2} \tag{48}$$

$$\mathrm{i}\left\{a_m, \tilde{\chi}_0\right\} = \tfrac{1}{2}\left(m + \tfrac{1}{2}\right) a_m . \tag{49}$$

The oscillator particle numbers $N_m = a_m^* a_m$ are all conserved quantities, so the model has an infinite number of conservation laws, but unfortunately these conserved quantities do not have vanishing Poisson brackets among themselves as required for an exactly solvable model [5, 6]. In fact Poisson bracket for any two such oscillator numbers is given by

$$\begin{aligned} \mathrm{i}\{N_m, N_n\} = \frac{1}{N}&\left(\sqrt{m+1}a_m^* a_{m+1} - \sqrt{m}a_{m-1}^* a_m\right)\left(\sqrt{n+1}a_{n+1}^* a_n - \sqrt{n}a_n^* a_{n-1}\right) \\ &+ m \leftrightarrow n . \end{aligned} \tag{50}$$

5. Quantization

The next task now would be to try to quantize the system. In doing this we immediately run into difficulties. Although the Fourier modes of the field ϕ_3 obey the simple equations of motion (8), and have the simple Poisson brackets (46)–(49) with the field bilinears, Poisson brackets among the Fourier modes themselves have the rather complicated forms (41) and (42). Although these two equations are quite well-defined as classical Poisson brackets, there is some ambiguity in the operator ordering of the commutators corresponding to these Poisson brackets. One can easily check that the most obvious choises for the required ordering are not consistent with the following crucial property of commutators

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right] \tag{51}$$

in the sense that the commutator of a product of Fourier modes of ϕ_3 would not necessarily have vanishing Poisson–Dirac brackets with the constraint (40). Even if some operator ordering consistent with all the requirements were found, it would not be clear how to proceed further since we do not know the Hilbert space representation of the quantum analogue of the algebra (41) and (42). The Fourier modes cannot be interpreted as quasi-particle annihilation operators since they are not independent, being subject to the constraint (40). The would-be quasi-particle numbers N_m, though conserved, cannot be specified arbitrarily even at the classical level since they have non-vanishing Poisson brackets given by equation (50). From the above, it appears that a direct quantization of the classical solution is not such an easy task. We should remember, however, that the many-body problem corresponding to this field theory is moderately easy to solve. As explained earlier, the energies and their precise degrees of degeneracy for N bosons of mass $\mu = \frac{1}{2}$ each are given by equation (4). In the rest of this paper, we shall show that equation (4) can be obtained in the semiclassical limit of a system of a large, fixed number N of particles near its ground state.

Before doing this, it may be useful to make the following remark. It might appear at first glance that the Poisson–Dirac algebra for the field ϕ_3, as given by equations (35) and (36), reduces in the large-N limit to the Poisson algebra for a free, unconstrained field as a result of the fact that all the extra terms in the algebra have an explicit factor of $1/N$ in front of them. This is, however, not true since some expectation values in N-particle states of the products of the field operators which the factor $1/N$ multiplies are themselves of first order in N, thus giving N-independent corrections to physical quantities. We shall therefore try to construct another set of operators whose commutator algebra would have a simple large-N limit. This will eventually explain the absence of the $k = 1$ term from the summation on the right hand side of equation (4). Such a term would be present if the large-N limit of equations (35) and (36) coincided with the corresponding relations for a free, unconstrained scalar field.

6. The raising/lowering operators

Since we are interested here in the spectrum of excitations of a system of a fixed number of particles, it is natural to look for operators that create or destroy such excitations without changing the particle number. The simplest such operators are field bilinears in the form

$$A = \langle \hat{A}(x, \partial_x) \rangle_{\phi_3} = \sum_{m,n} c_{mn} a_m^* a_n \tag{52}$$

where $\hat{A}(x, \partial_x)$ is some function of x and ∂_x. From a classical point of view, these quantities have zero Poisson brackets with the particle number N, as can be easily checked using equation (74). We shall be interested in those operators which lower or raise the energy of the states by a definite amount. The general form of a lowering operator A_m is

$$A_m = \sum_{k=0}^{\infty} \alpha_{m,k} a_m^* a_{m+k} \tag{53}$$

where $\alpha_{m,k}$ are arbitrary constants. Making use of equation (46), we can easily see that Poisson brackers of A_m with the Hamiltonian H is given by

$$\mathrm{i}\,\{A_m, H\} = m\omega A_m \tag{54}$$

which means that A_m lowers the energy by $m\omega$ units, while its Hermitian conjugate A_m^* raises the energy by the same amount. This is valid for any choice of the coefficients $\alpha_{m,k}$, but the requirement of Bose symmetry picks out one operator A_m, unique to within a multiplicative factor, for each positive integer value of m. This can be seen as follows. Since we are dealing with bosons, the operators A_m^* which create the the quasi-particles should commute, and the operators A_m which destroy the quasi-particles should also commute. Now Poisson bracket for any two operators A_m and A'_n of the form (53) can be easily calculated using equations (41) and (42). The result is

$$\mathrm{i}\left\{A_m, A'_n\right\} = \sum_k \left(\alpha_{m,k}\alpha'_{n,k+m} - \alpha'_{n,k}\alpha_{m,k+n}\right) a_k^* a_{k+m+n}$$

$$+ \frac{1}{N}\left(\sum_k \beta_{mk} a_k^* a_{k+m-1}\right)\left(\sum_l \gamma'_{nl} a_l^* a_{l+n+1}\right)$$

$$- \frac{1}{N}\left(\sum_k \gamma_{mk} a_k^* a_{k+m+1}\right)\left(\sum_l \beta'_{nl} a_l^* a_{l+n-1}\right) \tag{55}$$

where

$$\beta_{mk} = \alpha_{m,k}\sqrt{m+k} - \alpha_{m,k-1}\sqrt{k} \tag{56}$$

$$\gamma_{mk} = \alpha_{m,k}\sqrt{m+k+1} - \alpha_{m,k+1}\sqrt{k+1} \tag{57}$$

with similar relations for β' and γ' in terms of α'. When $m \neq n$ the bracket vanishes only when each of the three sums on the right hand side of equation (91) vanishes separately since these sums have different structures. To make the third term vanish, we must have either $\beta'_{nk} = 0$ or $\gamma_{mk} = 0$ for all values of k. The first condition cannot be satisfied for all values of k. The alternative condition, that of the vanishing of the γ-coefficients, gives

$$\alpha_{m,k} = C_m\sqrt{\frac{(m+k)!}{k!}} \tag{58}$$

where $C_m = \alpha_{m,0}$ is a possibly m-dependent normalization constant. If we now choose the coefficients $\alpha'_{n,k}$ in the same way, namely

$$\alpha'_{n,k} = C_n\sqrt{\frac{(n+k)!}{k!}} \tag{59}$$

the whole Poisson bracket for A_m and A'_n vanishes. Hence the required operators have the form

$$A_m = C_m \sum_{k=0}^{\infty} \sqrt{\frac{(m+k)!}{k!}} a_k^* a_{k+m}\,. \tag{60}$$

Putting $m = 0$ in equation (60), we see that $A_0 = C_0 N$, and hence A_0 is proportional to the particle number, while putting $m = 1$ gives $A_1 = 0$ by the constraint (31). Hence the actual quasi-particle creation and destruction operators are A_m and A_m^* with $m \geq 2$.

It is not difficult to see that

$$A_m = C_m \left\langle a^m \right\rangle_{\phi_3} \qquad A_m^* = C_m^* \left\langle (a^\dagger)^m \right\rangle_{\phi_3} \tag{61}$$

where a and $a^\dagger$ are given by equation (15), and are in fact the one-particle, first-quantized harmonic oscillator lowering and raising operators. Now, making use of the definitions (15) for a and $a^\dagger$, we can write equation (39) which gives the Poisson–Dirac bracket for any two dynamical variables A and B in the form

$$\{A, B\}_D = \{A, B\} + \left\{A, \langle a\rangle_{\phi_3}\right\}\frac{\mathrm{i}}{N}\left\{\langle a^\dagger\rangle_{\phi_3}, B\right\} - \left\{A, \langle a^\dagger\rangle_{\phi_3}\right\}\frac{\mathrm{i}}{N}\left\{\langle a\rangle_{\phi_3}, B\right\} \tag{62}$$

This form of the relation is more useful for our purposes since for any two one-particle operators $\hat{A}$ and $\hat{B}$, we have the following canonical Poisson bracket:

$$\mathrm{i}\{\langle\hat{A}\rangle_{\phi_3}, \langle\hat{B}\rangle_{\phi_3}\} = \langle[\hat{A}, \hat{B}]\rangle_{\phi_3}. \tag{63}$$

In particular, for powers of the operators a and $a^\dagger$, we have

$$\begin{aligned} &\left[a^m, a^n\right] = 0 \\ &\left[a^m, (a^\dagger)^n\right] = \sum_{k=1}^{\min(m,n)} \frac{1}{k!} \frac{m!n!}{(m-k)!(n-k)!} (a^\dagger)^{m-k} a^{n-k}. \end{aligned} \tag{64}$$

We shall now make the following choice of the normalization constants C_m

$$C_m = 1/\sqrt{N\,m!}$$

hence

$$A_m = \frac{1}{\sqrt{N\,m!}} \langle a^m\rangle_{\phi_3}. \tag{65}$$

Equations (63)–(65) may now be used to derive the following relation valid for $m \leq n$:

$$\begin{aligned} \mathrm{i}\left\{A_n, A_m^*\right\} &= \delta_{mn} + \frac{1}{\sqrt{N}}(1-\delta_{mn})\sqrt{\frac{n!}{m!\,(n-m)!}} A_{n-m} \\ &\quad + \frac{1}{N}\left(\sum_{k=1}^{m-1} \frac{\sqrt{n!\,m!}\,B_{n-m}^{(k)}}{k!\,(m-k)!\,(n-m+k)!} - \sqrt{nm}\,A_{n-1}A_{m-1}^*\right) \end{aligned} \tag{66}$$

where

$$B_l^{(k)} = \langle (a^\dagger)^k a^{l+k}\rangle = \sum_{j=k}^{\infty} \sqrt{\frac{j!\,(j+l)!}{(j-k)!}}\, a_j^* a_{j+l}. \tag{67}$$

The form in which equation (67) has been written may not be the simplest, but it will be more convenient for our purpose later.

7. The large-N limit

We have been able in the previous section to construct a maximal sequence of commuting operators A_m that would lower the energy by definite amounts, while not changing the particle number, with their Hermitian conjugates raising the energy but still not affecting the particle number. We still, nevertheless, have two problems to deal with. First, the construction of the previous section has been a classical one, based on the Poisson–Dirac brackets of the dynamical variables, and not addressing the problem of operator ordering. Then the Poisson–Dirac brackets (66) are still very complicated, and the algebra does not even close since the operators $B_{n-m}^{(k)}$ appear on the right hand side of equation (66). Hence the introduction of the A-operators seems at the first glance to have worsened the situation, leading to a more complicated algebra than that of the Fourier modes given by equations (41) and (42). The truth of the matter is that equation (66) has a very simple limit when the number of particles N becomes very large. To see this, let us try to see how the different terms in the equation behave when N becomes very large, with the system not very far from its ground state. In this case

$$N \sim N_0 = a_0^* a_0$$

hence

$$a_0 \sim \mathrm{O}(N^{1/2})$$

while

$$a_n \sim \mathrm{O}(N^0) \qquad \text{for } n \neq 0 .$$

Now A_m has the form

$$A_m = \frac{1}{\sqrt{N\,m!}} \left(\sqrt{m!} a_0^* a_m + \cdots \right)$$

hence

$$A_m \sim \mathrm{O}(N^0) .$$

Also

$$B_l^{(k)} \sim \mathrm{O}(N^0)$$

since the lowest order Fourier mode it contains is a_k^* as we can see from its defining equation (67). This means that for $m \leq n$

$$\mathrm{i}\left\{A_n, A_m^*\right\} = \delta_{mn} + \frac{1}{\sqrt{N}}(1-\delta_{mn})\sqrt{\frac{n!}{m!\,(n-m)!}}A_{n-m} + \mathrm{O}(N^{-1}). \tag{68}$$

This relation is also valid in the quantum theory as a commutation relation of the two operators A_m and A_n^* since all operator-ordering ambiguities are at least of first order in N^{-1}. We can now define the new set of operators $\tilde{A}_m$ by the relation

$$\tilde{A}_m = A_m - \frac{1}{\sqrt{N}} \sum_{k=2}^{\mathrm{Int}[(N+1)/2]} \sqrt{\frac{m!}{k!\,(m-k)!}} A_k A_{m-k} \tag{69}$$

where $\mathrm{Int}[(N+1)/2]$ is the integer part of $(N+1)/2$. A short calculation shows that the commutation relations for these new operators are

$$\left[\tilde{A}_m, \tilde{A}_n\right] = 0 \tag{70}$$

$$\left[\tilde{A}_m, \tilde{A}_n^*\right] = \delta_{mn} + \mathrm{O}(N^{-1}) \tag{71}$$

along with

$$\left[\tilde{A}_m, H\right] = m\omega\tilde{A}_m . \tag{72}$$

Hence in the limit of large N, these operators behave as the raising and lowering operators for an infinite set of independent harmonic oscillators such as the Fourier modes of a compact free field, and the states of the N-particle system can be labelled by the number of quasiparticles in each oscillator modes. These numbers n_m are the eigenvalues of the operators

$$\tilde{N}_m = \tilde{A}_m^* \tilde{A}_m \tag{73}$$

which commute among themselves, and with the Hamiltonian. We have seen earlier that the relevant values of m are the integers ≥ 2 since A_0 is proportional to the total particle number N, while A_1 and A_1^* vanish identically on account of the constraint (40). The operators A_2 and A_2^* are in fact related to $\tilde{\chi}_\pm$ by

$$A_2 = \sqrt{\frac{2}{N}}\,\tilde{\chi}_+ \qquad A_2^* = \sqrt{\frac{2}{N}}\,\tilde{\chi}_- . \tag{74}$$

Let us now define the ground state $|0\rangle$ of the N-particle system by

$$\begin{aligned} \hat{P}_N|0\rangle &= 0 \\ \hat{N}|0\rangle &= N|0\rangle \\ A_m|0\rangle &= 0 \qquad \text{for } m \geq 2 \end{aligned} \tag{75}$$

where

$$\hat{P}_N = \langle -\mathrm{i}\partial_x \rangle_\phi \qquad \hat{N} = \langle 1 \rangle_\phi .$$

Let E_0 be the energy of this state. We can now define an excited N-particle state of total momentum p_N and quasiparticle numbers $n_2, n_3, \ldots$ by

$$\hat{P}_N\,|p_N; n_2, n_3, \ldots\rangle = p_N\,|p_N; n_2, n_3, \ldots\rangle \tag{76}$$

$$\hat{N}_m\,|p_N; n_2, n_3, \ldots\rangle = n_m\,|p_N; n_2, n_3, \ldots\rangle . \tag{77}$$

This state can be obtained from the ground state $|0\rangle$ by repeated application of the quasiparticle creation operators and the momentum-changing operator

$$|p_N; n_2, n_3, \ldots\rangle = \mathrm{e}^{-\mathrm{i}\hat{x}_0 p_N} \prod_{m=2}^{\infty} \left(A_m^*\right)^{n_m} |0\rangle \tag{78}$$

where

$$\hat{x}_0 = \frac{\langle x \rangle_\phi}{N}$$

is the canonical conjugate of $\hat{P}_N$, which means that

$$\left[\hat{x}_0, \hat{P}_N\right] = \mathrm{i} \tag{79}$$

hence

$$\left[\hat{x}_0, \hat{H}\right] = \frac{\mathrm{i}2\hat{P}_N}{N} . \tag{80}$$

Using equations (72) and (80), we can easily see that the energy of the state defined by equations (76) and (77) is given by

$$E\,(p_N; n_1, n_2, \ldots) = E_0 + \frac{p_N^2}{N} + \omega \sum_{k=2}^{\infty} k\, n_k \tag{81}$$

which is precisely the value given by the exact N-body relation (4). This is the basic result of this work.

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